An Uncertainty Principle for Ensembles of Oscillators Driven by Common Noise

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We discuss control techniques for noisy self-sustained oscillators with a focus on reliability, stability of the response to noisy driving, and oscillation coherence understood in the sense of constancy of oscillation frequency. For any kind of linear feedback control—single and multiple delay feedback, linear frequency filter, etc.—the phase diffusion constant, quantifying coherence, and the Lyapunov exponent, quantifying reliability, can be efficiently controlled but their ratio remains constant. Thus, an "uncertainty principle" can be formulated: the loss of reliability occurs when coherence is enhanced and, vice versa, coherence is weakened when reliability is enhanced. Treatment of this principle for ensembles of oscillators synchronized by common noise or global coupling reveals a substantial difference between the cases of slightly non-identical oscillators and identical ones with intrinsic noise.

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Collective phenomena in ensembles of dynamical systems can manifest complex behavior and self-organization, which are not possible for unitary systems with arbitrary level of complexity. For biological systems, they extend from the simplest collective dynamics of bacteria [1] or higher organisms [2] to the activity of neural tissue [3, 4]. Similarly, in technology, essentially collective phenomena vary from plain synchronization effects [5, 6] to the phenomena laying in the basis of operation of artificial neural networks [3, 7]. Although these phenomena are inherent to large ensembles, their features are determined by characteristics which make physical sense also for unitary systems [6]. Indeed, synchronization (and clustering) in ensembles and their susceptibility to control are influenced by their individual robustness properties. "Reliability" [8] (or "consistency" [9]), i.e. the stability of the system response to noisy driving, and coherence [6, 10], i.e. the constancy of the oscillation frequency in time, are the principal ones.

Mathematically, the natural quantifier of reliability is the Lyapunov exponent (LE, λ) measuring the exponential decay rate of perturbations of the system response [6, 11, 12]. Oscillators are more reliable for a larger negative LE. Coherence can be quantified by the diffusion constant (DC, D) of the oscillation phase $\varphi(t)$, which grows non-uniformly in time due to either noise or chaotic dynamics; $\langle(\varphi(t)-\langle\dot{\varphi}\rangle t)^2\rangle \propto Dt$. These quantifiers are immediately relevant for the synchronization phenomenon in ensembles of oscillators subject to common noise [13, 14] or global coupling [5, 15, 16].

In this Letter, we report how the reliability and coherence of a noisy limit-cycle oscillator can be efficiently controlled by a general linear feedback. However, the ratio of LE and DC is shown to be independent of the noise strength and feedback parameters. The persistence of this ratio can be formulated as an *uncertainty principle*. Finally, the important implications of this principle for controlling collective dynamics of ensembles are revealed, and conclusions are drawn.

Let us consider an N-dimensional limit-cycle oscillator subject to small linear delayed feedback and weak noise:

$$\dot{x}_i = F_i(\mathbf{x}) + a\, z_i(t) + B_i(\mathbf{x}) \circ \xi(t), \qquad (1)$$

where $i = 1, 2, ..., N$, a is the feedback strength, the feedback term

$$z_i(t) = \int\limits_0^{+\infty} \sum_{j=1}^{N} G_{ij}(t_1)\, x_j(t-t_1)\, dt_1\,,$$

$G_{ij}(t)$ is the Green's function, different $G_{ij}(t)$ can feature single or multiple time delayed linear feedback (particular forms of this function are specified in the text below) [10, 17], linear frequency filter [18], etc.; "$\circ$" indicates the Stratonovich form of equation, $\xi(t)$ is white Gaussian noise: $\langle\xi\rangle = 0$, $\langle\xi(t)\xi(t')\rangle = 2\delta(t-t')$.

For weak noise and feedback the dynamics can be described within the framework of the phase description up to the leading order of accuracy [19, 20]:

$$\dot{\varphi} = \Omega_0 + a \int\limits_0^{+\infty} \sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}(t_1) H_{ij}\big(\varphi(t-t_1), \varphi(t)\big)\, dt_1 \\ + \varepsilon f\big(\phi(t)\big) \circ \xi(t), \qquad (2)$$

where Ω_0 is the natural frequency of the oscillator, f is a 2π-periodic function featuring the sensitivity of the phase to noise, ε is the noise amplitude, $H_{ij}(\psi,\varphi)$ is the increase of the phase growth rate created by the feedback term $x_j(\psi)$ acting on the variable $x_i(\varphi)$. The phase of the noise-free oscillator grows uniformly and its shifts neither decay nor grow in time. Noise creates irregularity of the phase growth rate, measured by the phase diffusion constant D (DC): $\big\langle(\varphi(t)-\langle\varphi(t)\rangle)^2\big\rangle \propto D\,t$. Additionally, noise results in convergence of trajectories, phase shifts decay, and the exponential rate of this decay is measured by the Lyapunov exponent λ (LE). Laborious

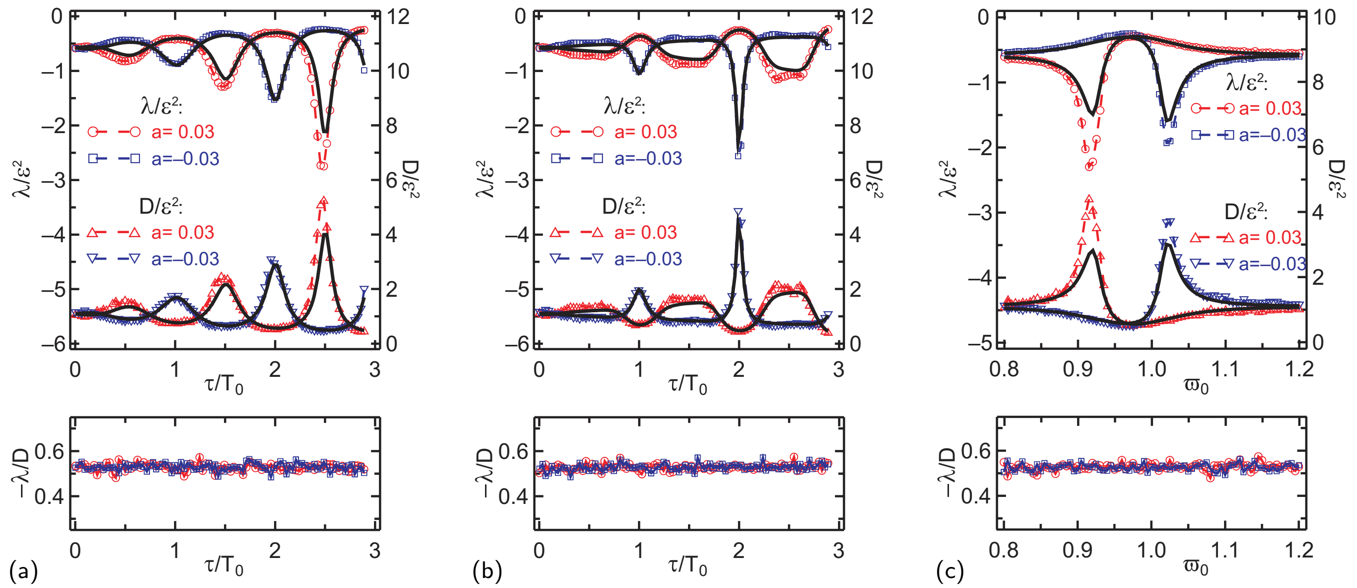

FIG. 1: (Color online) Dependencies of the Lyapunov exponent λ and the diffusion constant D (upper and lower graphs in the upper row of plots, respectively) on the feedback/filter parameters for the van der Pol oscillator with $\mu = 0.7$ subject to Gaussian white noise of strength $\varepsilon = 0.05$. (a) "Simple" delayed feedback [Eq. (8)]. (b) Multiple time delayed feedback [Eq. (9)]. (c) Linear frequency filter [Eq. (10)] with $\alpha = 0.1$. The oscillation period of the control-free noiseless system $T_0 \approx 2\pi/0.97$; the feedback strength ($a = 0.03$ and $a = -0.03$) is specified in plots. Solid black lines present analytical dependencies [Eqs. (3)–(5)] with $\langle f^2 \rangle \approx 0.5464$ and $\langle (f')^2 \rangle \approx 0.5775$. The ratio between the Lyapunov exponent and diffusion constant is constant and obeys Eq. (6) up to the calculation accuracy (see the lower row of plots).

but straightforward evaluation (non-general versions of which can be found in [10, 17, 18, 21]) yields

$$\Omega = \Omega_0 + a\int_0^{+\infty} \sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}(t) h_{ij}(-\Omega t)\, dt, \qquad (3)$$

$$D = \frac{2\varepsilon^2 \langle f^2 \rangle_\varphi}{\left(1 + a \int\limits_0^{+\infty} t \sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} G_{ij}(t) h'_{ij}(-\Omega t)\, dt\right)^2} = 2\varepsilon^2 \langle f^2 \rangle_\varphi \left(\frac{\partial \Omega}{\partial \Omega_0}\right)^2, \qquad (4)$$

$$\lambda = -\frac{\varepsilon^2 \langle (f')^2 \rangle_\varphi}{\left(1 + a \int\limits_0^{+\infty} t \sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} G_{ij}(t) h'_{ij}(-\Omega t)\, dt\right)^2} = -\varepsilon^2 \langle (f')^2 \rangle_\varphi \left(\frac{\partial \Omega}{\partial \Omega_0}\right)^2, \qquad (5)$$

where $\langle ... \rangle_\varphi \equiv (2\pi)^{-1}\int_0^{2\pi} ... d\varphi$, $h_{ij}(\psi) := \langle H_{ij}(\varphi + \psi, \varphi)\rangle_\varphi$, and the prime denotes derivative. Notice, for any kind of linear feedback the ratio

$$\frac{-\lambda}{D} = \frac{\langle (f')^2 \rangle}{2\langle f^2 \rangle} \qquad (6)$$

is independent of parameters of the feedback.

The modified formulation of the result (6) was revealed also for a certain class of systems where the phase cannot be well defined. Specifically, in systems below a Hopf bifurcation, noise can induce oscillatory motion [22] for which the phase is not well defined. For slightly perturbed periodic oscillations the autocorrelation function $C_{jj}(s) := \langle x_j(t) x_j(t+s)\rangle$ decays exponentially, $C_{jj}(s) = c_j(s)\exp(-D|s|)$, where function $c_j(s)$ oscillates but neither grows nor decays asymptotically. One can introduce an alternative quantifier of the irregularity of oscillations, the correlation time $t_{\mathrm{corr}} := C_{jj}^{-1}(0)\int_0^\infty |C_{jj}(s)| ds$ and find $t_{\mathrm{corr}} \propto 1/D$ for perturbed periodic oscillations. For noise-induced oscillations, t_{corr} is an appropriate quantifier, while D cannot be evaluated. In Refs. [22], for the van der Pol oscillator slightly below a Hopf bifurcation, the product $t_{\mathrm{corr}}|\lambda|$ was reported to be independent of the feedback with time-delay(s), although the feedback significantly changed t_{corr} and λ. This is equivalent to the result (6).

The validity of our findings (3)–(6) can be underpinned with the results of numerical simulation for noisy van der Pol oscillator

$$\dot{x} = y, \quad \dot{y} = \mu(1 - 4x^2)y - x + a z(t) + \varepsilon\, \xi(t). \qquad (7)$$

Here μ describes closeness to the Hopf bifurcation point, the phase $\varphi = -\arctan(y/x)$ (for $\mu \ll 1$ the limit cycle is: $x = \cos\varphi$, $y = -\sin\varphi$). We consider 3 possible cases: (1) a "simple" delayed feedback with delay time τ;

$$z^{(1\tau)}(t) = 2\big(y(t-\tau) - y(t)\big), \qquad (8)$$

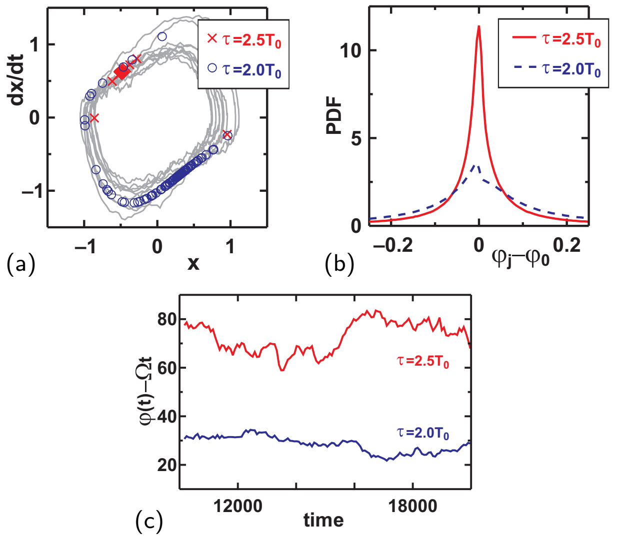

FIG. 2: (Color online) Ensemble of 100 van der Pol oscillators (7) with nonidentical frequencies subject to the simple delayed feedback (8). The distribution of frequencies is the gaussian one centered at 1 with standard deviation 0.001, feedback strength $a = 0.03$, noise amplitude $\varepsilon = 0.1$. (a) The gray line plots trajectory of one oscillator, the symbols plot snapshots of the ensemble for two specified values of the delay time (check these values in Fig. 1a). (b) The distribution of the phase deviations from the value φ_0 corresponding to the instantaneous ensemble-mean state. (c) The deviation of the phase from its mean growth for an oscillator (the offset of the vertical axis is arbitrary).

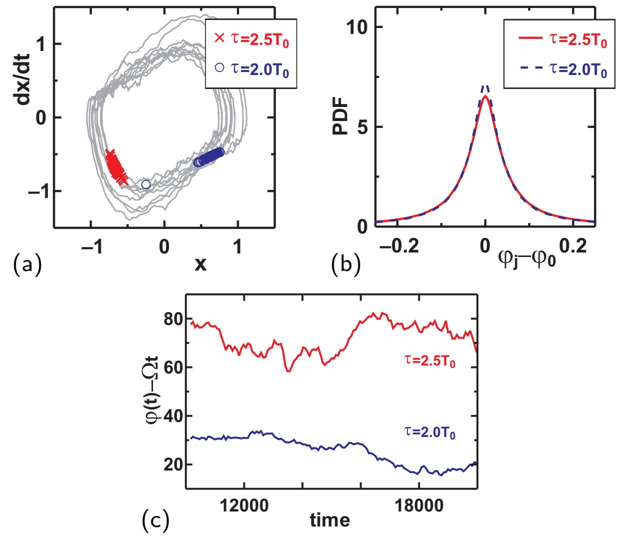

FIG. 3: (Color online) Ensemble of 100 van der Pol oscillators (7) subject to the simple delayed feedback (8), common noise $\varepsilon = 0.1$ and intrinsic noise $\varepsilon_{\rm int} = 0.005$. For details see caption to Fig. 2.

which yields

$$\hat{\mathbf{G}}^{(1\tau)}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 2\big(\delta(t-\tau) - \delta(t-0)\big) \end{bmatrix}$$

(here we explicitly indicate that $\int_0^{+\infty}\delta(t-0)dt = 1$).
(2) a multiple time delayed feedback;

$$z^{(\mathrm{m}\tau)}(t) = 2\sum_{n=0}^{+\infty} R^n\big(y(t-(n+1)\tau) - y(t-n\tau)\big), \quad (9)$$

where $|R| < 1$, which yields

$$\hat{\mathbf{G}}^{(\mathrm{m}\tau)}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 2\sum\limits_{n=0}^{+\infty} R^n\big(\delta(t-(n+1)\tau) - \delta(t-n\tau-0)\big) \end{bmatrix}.$$

(3) a linear frequency filter;

$$z^{(\mathrm{lff})}(t) = 2\dot{u}(t), \qquad \ddot{u} + \alpha\dot{u} + \tilde{\omega}_0^2 u = \alpha x(t), \quad (10)$$

which yields

$$\hat{\mathbf{G}}^{(\mathrm{lff})}(t) = \begin{bmatrix} 0 & 0 \\ 2\alpha e^{-\alpha t/2}(\cos\omega_0 t - \frac{\alpha}{2\omega_0}\sin\omega_0 t) & 0 \end{bmatrix},$$

$\omega_0 = \sqrt{\tilde{\omega}_0^2 - (\alpha/2)^2}$.

For $\mu \ll 1$, the van der Pol oscillator possesses a circular limit-cycle and $h_{21}(\varphi) = -(1/2)\cos\varphi$, $h_{22}(\varphi) = (1/2)\sin\varphi$, $f(\varphi) = -\sin\varphi$. The analytical results derived with the above formulae match well the results of numerical simulation [10, 17, 18], as can be well seen in Fig. 1. Remarkably, the ratio between LE and DC is constant with a good accuracy even when they vary by one order of magnitude, in agreement with Eq. (6).

Constancy of the ratio (6) can be formulated as a kind of "*uncertainty principle*" for the linear feedback control techniques:

The reliability of a noisy oscillator can be significantly enhanced (by means of a weak linear feedback) but at the price of the loss of its coherence and, vice versa, the coherence can be significantly enhanced but with the loss of reliability.

For ensembles of uncoupled oscillators driven by common noise we know, that the imperfectness of identity of oscillators or intrinsic noise leads to imperfectness of synchronization. The characteristic spreading of states $\propto 1/\sqrt{-\lambda}$ [14]. That is, higher reliability leads to stronger synchrony of oscillations. Thinking of employment of control techniques we should distinguish two "pure" situations: (i) slightly non-identical oscillators driven by identical noise and (ii) identical oscillators with small intrinsic noise, individual for each oscillator.

For an ensemble of non-identical oscillators, the dispersing of phases is owed to the mismatch of natural frequencies. This mismatch is nearly unaffected by small feedback, while a synchronizing action measured by LE is influenced by the feedback. Hence, the dispersion of states is perceptive to the feedback control. For instance, one can see in Fig. 2 that the ensemble is well syn-

chronized (Fig. 2a,b) when the phase diffusion is large (Fig. 2c), and, on the contrary, for a small diffusion (enhanced coherence) we observe a much poorer synchrony.

For an ensemble of identical oscillators with intrinsic noise, the situation is different. The dispersion is owed to the mutual diffusion of oscillator phases created by intrinsic noise. Similarly to [14], we can find $\Delta\varphi \propto \sqrt{D_{\rm int}}/\sqrt{-\lambda}$, where $D_{\rm int}$ characterizes the diffusion due to intrinsic noise. The diffusion owing to intrinsic noise is expected to be subjected to the effect of the feedback in the same manner as the total diffusion. Hence, we expect the feedback to not influence the phase dispersion, $\Delta\varphi \propto \sqrt{D_{\rm int}}/\sqrt{-\lambda} \approx const$. Indeed, in Fig. 3 one can see that the distribution of phase deviations is tolerant to feedback control. In this case, one can employ the feedback control techniques to control the coherence without the loss of synchrony.

Summarizing, we have discovered the universal *uncertainty principle*, which is valid for general noisy limit-cycle oscillators subject to a general linear feedback control and proved it both analytically and numerically. Mathematically, this uncertainty principle takes the form of Eq. (6); the ratio of the Lyapunov exponent (λ), measuring reliability (*or* the response stability, [8, 9]), and the diffusion constant (D), measuring coherence (*or* the constancy of the instantaneous frequency in time), is independent of the noise strength and feedback parameters. That is the reliability of a noisy oscillator can be significantly enhanced by means of a relatively weak linear feedback, but at the price of the loss of its coherence, and vice versa, the coherence can be significantly enhanced but with the loss of reliability. The principle has an implication to practical issues of the control of synchronization in non-ideal ensembles of oscillators in nature and technology [1, 2, 4, 6, 13, 16]. For ensembles of weakly non-identical oscillators driven by common noise (cf Fig. 2) the enhancement of synchrony, achieved due to the reliability increase, results in a poorer coherence of each individual oscillator. Unsimilarly, for ensembles of identical oscillators with intrinsic noise (cf Fig. 3) the synchrony is not influenced by the change of reliability/coherence; therefore, coherence can be controlled without stray effects on the synchronization.

Notice, that ensembles of globally coupled oscillators in an asynchronous state immediately correspond to the case of uncoupled oscillators receiving common driving; this driving is the ensemble-mean value forcing an oscillator via the coupling term [15]. As long as collective modes vanish, the ensemble-mean value fluctuating about zero may be well considered as a stochastic process and one can speak of synchronization by common noise. (When the collective mode appears, it can be regular in time; synchronization by a regular signal drastically differs from that by a nonperiodic one, e.g. [12, 23].) Therefore, our findings are related to the fundamentals of synchronization in ensembles of globally coupled oscillators as well [16].

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